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Applications of Dynamical Algebra of Invariance of Integrodifferential Equations

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General results on the algebraic properties of integrodifferential equations are used to obtain coherent and squeezed states and Green functions for the matrix-differential models of condensed matter theory.

The study of algebraic properties of integrodifferential equations is a matter of general theoretical interest for anyone concerned with the most effective tool for the solution of pure and applied physical problems (Meleshco, 1986; Rustamov, 1987*a*).

The present work is devoted to the application of general statements on the algebraic properties of integrodifferential equations (Rustamov, 1987*a*) to get solutions of such traditional problems of quantum theory as the construction of coherent and squeezed states and Green functions for models of condensed matter physics of matrix-differential form. As an example of the effectiveness of the suggested scheme, the Kane model (Kane, 1957; Rustamov, 1988) of the energy spectra of electric charge carriers in cubic solids will be analyzed.

Let us consider Schrödinger equations with Hamilton operators matrix-differential form acting on the multidimensional vector space L_n. For instance,

$$H=\begin{pmatrix} h_{11}(x_1,\ldots,x_j;\partial/\partial x_1,\ldots,\partial/\partial x_j;\partial^2/\partial x_i\,\partial x_j,\ldots)\cdots & \\ \vdots & \\ h_{1n}(x_1,\ldots,x_j;\partial/\partial x_1,\ldots,\partial/\partial x_j;\partial^2/\partial x_i\,\partial x_j,\ldots)\cdots h_{nn}(\cdots) & \end{pmatrix},$$

$$(i,j=1,\ldots,l);\qquad (x_1,\ldots,x_l)\in\mathbb{R}^l;\qquad L_n\ni\Psi=\begin{pmatrix}\Psi_1(x_1,\ldots,x_l;t)\\ \vdots\\ \Psi_n(x_1,\ldots,x_l;t)\end{pmatrix}$$

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where the functions $\Psi_k(x_1, \ldots, x_l; t)$ belong to Hilbert space $(k=1, \ldots, n)$.

In many interesting cases the operator H does not depend on $(x_1, \ldots, x_l)$ and their symbols (Shubin, 1978) can be easily diagonalized (Kane, 1957; Rustamov, 1987*b*, 1988).

Thus, matrix-differential forms H will be represented as

$$\Delta \bar{H} \Delta^{-1} = \delta_{nk} \lambda_n(P_1, \ldots, P_j)$$

where Δ is a unitary diagonalizing operation, $\bar{H}$ is the symbols of operator H, δ_{nk} is the Kronecker symbol, and the λ_n are functions depending on the Fourier images of $\partial/\partial x_j$ $(j=1, \ldots, l)$.

Hence, instead of the original Schrödinger equations, one deals with the integrodifferential equations defined by $\lambda_n(P_1, \ldots, P_j)$ or their inverse Fourier images $\bar{\lambda}_n(\partial/\partial x_1, \ldots, \partial/\partial x_j)$.

Due to a general theorem presented in Rustamov (1987*a*), the Heizenberg-Weyl algebra of invariance of integrodifferential equations on the type

$$\left\{ i\frac{\partial}{\partial t} + \left[\sum_{j=1}^{l} \left(\frac{\partial^2}{\partial x_j^2} + b_j \right) \right]^{\gamma/\beta} \right\} \Psi(x_1, \ldots, x_l; t) = 0$$

may be represented by the operators

$$\frac{\partial}{\partial x_i}; \qquad x_i + \sigma t \left(\frac{\gamma}{\beta} \right) \frac{\partial}{\partial x_i} \left[\sum_{j=1}^{l} \left(\frac{\partial^2}{\partial x_j^2} + b_j \right) \right]^{(\gamma-\beta)/\beta}$$

and I, the identity transformation, where γ, β, σ, and b_j are some constants.

For example, in the case of the widely used Kane model, one gets the following linear combinations of such operators:

$$A_j(R_\nu) = -\frac{1}{2} \left(\frac{\partial}{\partial P_j} - \frac{2iR_\nu}{3\hbar} \frac{B^2 t P_j}{(\varepsilon_g^2/4 + \frac{2}{3}B^2 P^2)^{1/2}} \right) + P_j$$

$$A_j^+(R_\nu) = -\frac{1}{2} \left(\frac{\partial}{\partial P_j} - \frac{2iR_\nu}{3\hbar} \frac{B^2 t P_j}{(\varepsilon_g^2/4 + \frac{2}{3}B^2 P^2)^{1/2}} \right) - P_j \qquad (j=1, \ldots, l;\ l=3)$$

where B, ε_g, $\hbar$ are constants, $P^2 = \sum_j P_j P_j$, and R_ν is a constant which labels the energy bands (Rustamov, 1987*b*).

Such important problems of traditional physical interest as the construction of the coherent and squeezed states and Green functions may be directly solved using these Heizenberg-Weyl algebra operators (Perelomov, 1972; Man'ko, 1979).

Namely, the coherent states (interband ones, to be more precise) in the case of the Kane model are

$$|\alpha; R_\nu\rangle = \left(\frac{-2}{\pi} \right)^{3/4} \exp\left\{ \frac{iR_\nu t}{\hbar} \left(\frac{\varepsilon_g^2}{4} + \frac{2B^2P^2}{3} \right)^{1/2} + P^2 + \sum_j \left(\frac{\alpha_j^2 - |\alpha_j|^2}{2} - 2\alpha_j P_j \right) \right\}$$

such as

$$A_j(R_\nu)|\alpha;R_\nu\rangle=\alpha_j|\alpha;R_\nu\rangle$$

or

$$|\alpha;R_\nu\rangle=\mathscr{D}(\alpha)|0;R_\nu\rangle=\prod_j\exp[\alpha_jA_j^+(R_\nu)-\alpha_j^*A(R_\nu)]|0;R_\nu\rangle$$

where $A_j(R_\nu)|0;R_\nu\rangle=0$, and α_j are the complex eigenvalues of the operators $A_j(R_\nu)$.

The so-called squeezed states (Yuen, 1976; Walls, 1983) with the squeeze parameter ζ, $|\alpha,\zeta;R_\nu\rangle$, are given by

$$|\alpha,\zeta;R_\nu\rangle=\exp\left(\frac{3\,\mathrm{Re}^2[\zeta(\mu+\delta)^*]}{|\mu+\delta|^2\,\mathrm{Re}[(\mu+\delta)(\mu-\delta)^*]}\right)$$
$$\times\left(\frac{-2\,\mathrm{Re}[(\mu+\delta)(\mu-\delta)^*]}{\pi|\mu+\delta|^2}\right)^{3/4}$$
$$\times\exp\left\{\frac{\mu-\delta}{\mu+\delta}P^2-\frac{2}{\mu+\delta}\sum_j\zeta_jP_j+\frac{iR_\nu t}{\hbar}\left(\frac{\varepsilon_g^2}{4}+\frac{2B^2P^2}{3}\right)^{1/2}\right\}$$

where μ and δ satisfy the condition $\mu^2-\delta^2=1$ and are the mixing parameters of the single coherent mode with its phase conjugate (Yuen, 1976).

Using the explicit form of $|\alpha;R_\nu\rangle$, it is easy to obtain the Green functions of the problem. Namely

$$G(P',t';P,t)$$
$$=\pi^{-3}\int_{-\infty}^{\infty}\langle P',t'|\alpha;R_\nu\rangle\langle\alpha;R_\nu|P,t\rangle\,d^2\alpha$$
$$=\exp\left\{\frac{iR_\nu t'}{\hbar}\left(\frac{\varepsilon_g^2}{4}+\frac{2B^2P'^2}{3}\right)^{1/2}-\frac{iR_\nu t}{\hbar}\left(\frac{\varepsilon_g^2}{4}+\frac{2b^2P^2}{3}\right)^{1/2}\right.$$
$$\left.+P'^2+P^2-\sum_j\frac{(P_j-P_j')^2}{2}\right\}\delta(P-P')$$

where $\delta(P-P')=\prod_j\delta(P_j-P_j')$; $d^2\alpha=d(\mathrm{Re}\,\alpha)\,d(\mathrm{Im}\,\alpha)$.

It is easy to see that above functions satisfy the general conditions defined for these types of objects, for example, orthogonality and overcompleteness for the coherent states (Man'ko, 1979), etc.

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